

AN OPTIMAL BOUNDEDNESS ON WEAK \mathbb{Q} -FANO THREEFOLDS

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ABSTRACT. Let X be a terminal weak \mathbb{Q} -Fano threefold. We prove that $P_{-6}(X) > 0$ and $P_{-8}(X) > 1$. We also prove that the anti-canonical volume has a universal lower bound $-K_X^3 \geq 1/330$. This lower bound is optimal.

1. Introduction

A threefold X is said to be a terminal (resp. canonical) \mathbb{Q} -Fano threefold if X has at worst terminal (resp. canonical) singularities and $-K_X$ is ample, where K_X is a canonical Weil divisor on X . X is called a *terminal weak \mathbb{Q} -Fano threefold* if X has at worst terminal singularities and $-K_X$ is nef and big.

We are interested in a conjecture of Miles Reid [8, Section 4.3] which says that $P_{-2}(X) > 0$ for almost all \mathbb{Q} -Fano 3-folds. There are already several known examples with $P_{-2} = 0$ by Iano-Fletcher [4] and Altinok and Reid [1]. Another question that we are interested in is the boundedness of \mathbb{Q} -Fano 3-folds, which is equivalent to the boundedness of the anti-canonical volume $-K_X^3$. Kawamata [5] first showed the boundedness of $-K^3$ for terminal \mathbb{Q} -Fano 3-folds with Picard number 1. Then Kollár, Miyaoka, Mori and Takagi [7] gave the boundedness for all canonical \mathbb{Q} -Fano 3-folds. Recently Brown and Susuki [2] proved a sharp lower bound of $-K^3$ for certain \mathbb{Q} -Fano 3-folds. However a practical lower bound of $-K^3$ for all \mathbb{Q} -Fano 3-folds is still unknown, which is another motivation of our paper.

Our main results are the following:

Theorem 1.1. *Let X be a terminal weak \mathbb{Q} -Fano 3-fold. Then*

- (i) $P_{-4} > 0$ with possibly one exception of basket of singularities;
- (ii) $P_{-6} > 0$ and $P_{-8} > 1$;
- (iii) $-K_X^3 \geq \frac{1}{330}$. Furthermore $-K_X^3 = \frac{1}{330}$ if and only if the virtual basket of singularities is

$$\left\{ \frac{1}{2}(1, -1, 1), \frac{1}{5}(1, -1, 2), \frac{1}{3}(1, -1, 1), \frac{1}{11}(1, -1, 2) \right\}.$$

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The lower bound $\frac{1}{330}$ is optimal due to the following example by Iano-Fletcher:

Example 1.2. ([4, Page 158]) The general hypersurface

$$X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$$

has $-K_X^3 = \frac{1}{330}$.

We now sketch our method of baskets and explain the idea of the proofs. Recall that Reid's Riemann-Roch formula describes the Euler characteristic by counting the contribution from virtual quotient singularities, which he calls *basket*. We remark that when either K_X or $-K_X$ is nef and big, then Euler characteristic is nothing but plurigenus or anti-plurigenus. Our method in [3] provides a synthetic way to recover baskets in terms of plurigenera. Even though one can not expect to recover baskets completely with limited information from plurigenera. However the possibility of baskets is finite when P_{-m} is small for small m .

The behavior of baskets in \mathbb{Q} -Fano case is somehow nicer. One reason is that $\chi(\mathcal{O}_X) = 1$. And thanks to many effective inequalities derived from the basket trick, we can prove that there are only finitely many baskets with given P_{-1} and P_{-2} (see 3.3). Furthermore we can give a complete list of those small anti-plurigenera formal baskets satisfying geometric constrains (2.1), (2.2) and (2.3). This allows us to prove our statements.

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2. Baskets of pairs and geometric inequalities

In this section, we would like to recall our method developed in [3], together with some geometrical inequalities which will be the core of our proof.

A *basket* B is a collection of pairs of integers (permitting weights) $\{(b_i, r_i) | i = 1, \dots, t; b_i \text{ coprime to } r_i\}$.¹ For simplicity, we will frequently write a basket in another way, say

$$B = \{(1, 2), (1, 2), (2, 5)\} = \{2 \times (1, 2), (2, 5)\}.$$

¹We may drop the assumption of coprime if we simply consider $\{(db, dr), *\}$ as $\{d \times (r, b), *\}$. These two baskets share all the same numerical properties in our discussion.

2.1. Reid's formula. Let X be a terminal weak \mathbb{Q} -Fano 3-fold. According to Reid [8], there is a basket of pairs

$$B_X = \{(b_i, r_i) | i = 1, \dots, t; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}.$$

such that, for all integer $n > 0$,

$$P_{-n}(X) = \frac{1}{12}n(n+1)(2n+1)(-K_X^3) + (2n+1) - l(-n)$$

where $l(-n) = l(n+1) = \sum_i \sum_{j=1}^n \frac{\overline{jb_i(r_i - jb_i)}}{2r_i}$ and $\overline{\cdot}$ means the smallest residue mod r_i .

The above formula can be rewritten as:

$$\begin{aligned} P_{-1} &= \frac{1}{2}(-K_X^3 + \sum_i \frac{b_i^2}{r_i}) - \frac{1}{2} \sum_i b_i + 3, \\ P_{-m} - P_{-(m-1)} &= \frac{m^2}{2}(-K_X^3 + \sum_i \frac{b_i^2}{r_i}) - \frac{m}{2} \sum_i b_i + 2 - \Delta^m \end{aligned}$$

where $\Delta^m = \sum_i (\frac{\overline{b_i m(r_i - b_i m)}}{2r_i} - \frac{b_i m(r_i - b_i m)}{2r_i})$ and $m \geq 2$.

Notice that all the anti-plurigenera P_{-n} can be determined by the basket B_X and $P_{-1}(X)$. This leads us to set the following definitions for *formal baskets*.

We recall some definitions and properties of baskets. Especially, we introduce the notion of packing. All details can be found in Section 4 of [3].

Suppose that $B = \{(b_i, r_i) | i = 1, \dots, t; 0 < b_i \leq \frac{r_i}{2}; b_i \text{ is coprime to } r_i\}$ is a basket. Let $n > 1$ be an integer. For each i , set $l_i := \lfloor \frac{nb_i}{r_i} \rfloor$ and define

$$\Delta_i^n := l_i b_i n - \frac{1}{2}(l_i^2 + l_i)r_i,$$

which can be shown to be a non-negative integer. Define $\Delta^n(B) = \sum_{i=1}^t \Delta_i^n$. One can verify that $\Delta^n(B) = \sum_i (\frac{\overline{b_i n(r_i - b_i n)}}{2r_i} - \frac{b_i n(r_i - b_i n)}{2r_i})$.

We set $\sigma(B) := \sum_i b_i$ and $\sigma'(B) := \sum_i \frac{b_i^2}{r_i}$.

Given a basket $B = \{(b_i, r_i) | i = 1, \dots, t\}$ and assume that $b_1 + b_2$ is coprime to $r_1 + r_2$, then we say that the new basket $B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \dots, (b_t, r_t)\}$ is a packing of B , denoted as $B \succ B'$. We call $B \succ B'$ a *prime packing* if $b_1 r_2 - b_2 r_1 = 1$. A composition of finite packings is also called a packing. So the relation " \succeq " is a partial ordering on the set of baskets.

2.2. Properties of packings. As we have proved in [3], a packing has the following properties:

Assume $B \succeq B'$. Then

- i. $\sigma(B) = \sigma(B')$ and $\sigma'(B) \geq \sigma'(B')$;

ii. For all integer $n > 1$, $\Delta^n(B) \geq \Delta^n(B')$;

2.3. Formal baskets. We call a pair (B, \tilde{P}_{-1}) a *formal basket* if B is a basket and \tilde{P}_{-1} is a non-negative integer. We write

$$(B, \tilde{P}_{-1}) \succ (B', \tilde{P}_{-1})$$

if $B \succ B'$.

We define some invariants of formal baskets. Considering a formal basket $\mathbf{B} = (B, \tilde{P}_{-1})$, define $\tilde{P}_{-1}(\mathbf{B}) := \tilde{P}_{-1}$, the volume

$$-K^3(\mathbf{B}) := 2\tilde{P}_{-1} + \sigma(B) - \sigma'(B) - 6$$

and $\tilde{P}_{-2}(\mathbf{B}) := 5\tilde{P}_{-1} + \sigma(B) - 10$. So one has

$$\tilde{P}_{-2}(\mathbf{B}) - \tilde{P}_{-1}(\mathbf{B}) = 2(-K^3(\mathbf{B}) + \sigma'(B)) + 2 - \sigma(B).$$

For all $m \geq 2$, we define the anti-plurigenus in an inductive way:

$$\tilde{P}_{-(m+1)} - \tilde{P}_{-m} = \frac{1}{2}(m+1)^2(-K^3(\mathbf{B}) + \sigma'(B)) + 2 - \frac{m+1}{2}\sigma - \Delta^{m+1}(B).$$

Notice that $\tilde{P}_{-(m+1)} - \tilde{P}_{-m}$ is an integer because $-K^3(\mathbf{B}) + \sigma'(B) = 2\tilde{P}_{-1} + \sigma(B) - 6$ has the same parity as that of $\sigma(B)$.

Now if $B = B_X$ for a terminal weak \mathbb{Q} -Fano 3-fold X and $\tilde{P}_{-1} = P_{-1}(X)$, then one can verify that $-K^3(\mathbf{B}) = -K_X^3$ and $\tilde{P}_{-m}(\mathbf{B}) = P_{-m}(X)$ for all $m \geq 2$.

2.4. Properties of packings (of formal baskets). By 2.2 and the above formulae, one can see the following immediate properties of formal baskets:

Assume $\mathbf{B} := (B, \tilde{P}_{-1}) \succeq \mathbf{B}' := (B', \tilde{P}_{-1})$. Then

- iii. $-K^3(\mathbf{B}) + \sigma'(B) = -K^3(\mathbf{B}') + \sigma'(B')$;
- iv. $-K^3(\mathbf{B}) \leq -K^3(\mathbf{B}')$;
- v. $\tilde{P}_{-m}(\mathbf{B}) \leq \tilde{P}_{-m}(\mathbf{B}')$ for all $m \geq 2$.

2.5. Canonical sequence of baskets. Next we recall the “canonical” sequence of a basket B . Set $S^{(0)} := \{\frac{1}{n} | n \geq 2\}$, $S^{(5)} := S^{(0)} \cup \{\frac{2}{5}\}$ and inductively for all $n \geq 5$,

$$S^{(n)} := S^{(n-1)} \cup \left\{ \frac{b}{n} \mid 0 < b < \frac{n}{2}, b \text{ coprime to } n \right\}.$$

Defined in this way, then each set $S^{(n)}$ gives a division of the interval $(0, \frac{1}{2}] = \bigcup_i [\omega_{i+1}^{(n)}, \omega_i^{(n)}]$ with $\omega_i^{(n)}, \omega_{i+1}^{(n)} \in S^{(n)}$. Let $\omega_{i+1}^{(n)} = \frac{q_{i+1}}{p_{i+1}}$ and $\omega_i^{(n)} = \frac{q_i}{p_i}$ with $\text{g.c.d.}(q_l, p_l) = 1$ for $l = i, i+1$. Then it is easy to see that $q_i p_{i+1} - p_i q_{i+1} = 1$ for all n and i (cf. [3, Claim A]).

Now given a basket $B = \{(b_i, r_i) | r = 1, \dots, t\}$, we would like to define new baskets $B^{(n)}(B)$. For each $B_i = (b_i, r_i) \in B$, if $\frac{b_i}{r_i} \in S^{(n)}$, then we set $B_i^{(n)} := \{(b_i, r_i)\}$. If $\frac{b_i}{r_i} \notin S^{(n)}$, then $\omega_{l+1}^{(n)} < \frac{b_i}{r_i} < \omega_l^{(n)}$ for some l . We write $\omega_l^{(n)} = \frac{q_l}{p_l}$ and $\omega_{l+1}^{(n)} = \frac{q_{l+1}}{p_{l+1}}$ respectively.

In this situation, we can unpack (b_i, r_i) to $B_i^{(n)} := \{(r_i q_l - b_i p_l) \times (q_{l+1}, p_{l+1}), (-r_i q_{l+1} + b_i p_{l+1}) \times (q_l, p_l)\}$. Adding up those $B_i^{(n)}$, we get a new basket $B^{(n)}(B)$. $B^{(n)}(B)$ is uniquely defined according to our construction and $B^{(n)}(B) \succ B$ for all n . Notice that $B = B^{(n)}(B)$ for n sufficiently large, e.g. for $n \geq \max\{r_i\}$.

In fact, we have

$$B^{(n-1)}(B) = B^{(n-1)}(B^{(n)}(B)) \succ B^{(n)}(B)$$

for all $n \geq 1$ (cf. [3, Claim B]). Therefore we have a chain of baskets:

$$B^{(0)}(B) \succ B^{(5)}(B) \succ \dots \succ B^{(n)}(B) \succ \dots \succ B.$$

The step $B^{(n-1)}(B) \succ B^{(n)}(B)$ can be achieved by a number of successive prime packings. Let $\epsilon_n(B)$ be the number of such prime packings.

We recall the following easy but essential properties.

Lemma 2.6. ([3, Lemma 4.15]) *For the sequence $\{B^{(n)}(B)\}$, the following statements are true:*

- (i) $\Delta^j(B^{(0)}(B)) = \Delta^j(B)$ for $j = 3, 4$;
- (ii) $\Delta^j(B^{(n-1)}(B)) = \Delta^j(B^{(n)}(B))$ for all $j < n$;
- (iii) $\Delta^n(B^{(n-1)}(B)) = \Delta^n(B^{(n)}(B)) + \epsilon_n(B)$.

It follows that $\Delta^j(B^{(n)}(B)) = \Delta^j(B)$ for all $j \leq n$ and

$$\epsilon_n(B) = \Delta^n(B^{(n-1)}(B)) - \Delta^n(B^{(n)}(B)) = \Delta^n(B^{(n-1)}(B)) - \Delta^n(B).$$

Moreover, given a formal basket $\mathbf{B} = (B, \tilde{P}_{-1})$, we can similarly consider $B^{(n)}(\mathbf{B}) := (B^{(n)}(B), \tilde{P}_{-1})$. It follows that

$$\tilde{P}_{-j}(B^{(n)}(\mathbf{B})) = \tilde{P}_{-j}(\mathbf{B}) \text{ for all } j \leq n.$$

Therefore we can realize the canonical sequence of formal baskets as an approximation of formal baskets via anti-plurigenera.

2.7. Solving formal baskets by anti-plurigenera.

We now study the relation between formal baskets and anti-plurigenera more closely. For a given formal basket $\mathbf{B} = (B, \tilde{P}_{-1})$, we begin by computing the non-negative number ϵ_n and $B^{(0)}, B^{(5)}$ in terms of \tilde{P}_{-m} . From the definition of \tilde{P}_{-m} we get:

$$\sigma(B) = 10 - 5\tilde{P}_{-1} + \tilde{P}_{-2},$$

$$\begin{aligned} \Delta^{m+1} &= (2 - 5(m+1) + 2(m+1)^2) + \frac{1}{2}(m+1)(2-3m)\tilde{P}_{-1} \\ &\quad + \frac{1}{2}m(m+1)\tilde{P}_{-2} + \tilde{P}_{-m} - \tilde{P}_{-(m+1)}. \end{aligned}$$

In particular, we have:

$$\begin{aligned}\Delta^3 &= 5 - 6\tilde{P}_{-1} + 4\tilde{P}_{-2} - \tilde{P}_{-3}; \\ \Delta^4 &= 14 - 14\tilde{P}_{-1} + 6\tilde{P}_{-2} + \tilde{P}_{-3} - \tilde{P}_{-4};\end{aligned}$$

Assume $B^{(0)}(B) = \{n_{1,r}^0 \times (1, r) | r \geq 2\}$. By Lemma 2.6, we have

$$\begin{aligned}\sigma(B) &= \sigma(B^{(0)}(B)) = \sum n_{1,r}^0; \\ \Delta^3(B) &= \Delta^3(B^{(0)}(B)) = n_{1,2}^0; \\ \Delta^4(B) &= \Delta^4(B^{(0)}(B)) = 2n_{1,2}^0 + n_{1,3}^0.\end{aligned}$$

Thus one gets $B^{(0)}$ as follows:

$$\begin{cases} n_{1,2}^0 = 5 - 6\tilde{P}_{-1} + 4\tilde{P}_{-2} - \tilde{P}_{-3} \\ n_{1,3}^0 = 4 - 2\tilde{P}_{-1} - 2\tilde{P}_{-2} + 3\tilde{P}_{-3} - \tilde{P}_{-4} \\ n_{1,4}^0 = 1 + 3\tilde{P}_{-1} - \tilde{P}_{-2} - 2\tilde{P}_{-3} + \tilde{P}_{-4} - \sigma_5 \\ n_{1,r}^0 = n_{1,r}^0, r \geq 5, \end{cases}$$

where $\sigma_5 := \sum_{r \geq 5} n_{1,r}^0$. A computation gives:

$$\epsilon_5 = 2 + \tilde{P}_{-2} - 2\tilde{P}_{-4} + \tilde{P}_{-5} - \sigma_5.$$

Therefore we get $B^{(5)}$ as follows:

$$\begin{cases} n_{1,2}^5 = 3 - 6\tilde{P}_{-1} + 3\tilde{P}_{-2} - \tilde{P}_{-3} + 2\tilde{P}_{-4} - \tilde{P}_{-5} + \sigma_5 \\ n_{2,5}^5 = 2 + \tilde{P}_{-2} - 2\tilde{P}_{-4} + \tilde{P}_{-5} - \sigma_5 \\ n_{1,3}^5 = 2 - 2\tilde{P}_{-1} - 3\tilde{P}_{-2} + 3\tilde{P}_{-3} + \tilde{P}_{-4} - \tilde{P}_{-5} + \sigma_5 \\ n_{1,4}^5 = 1 + 3\tilde{P}_{-1} - \tilde{P}_{-2} - 2\tilde{P}_{-3} + \tilde{P}_{-4} - \sigma_5 \\ n_{1,r}^5 = n_{1,r}^0, r \geq 5 \end{cases}$$

Because $B^{(5)} = B^{(6)}$, we see $\epsilon_6 = 0$ and on the other hand

$$\epsilon_6 = 3\tilde{P}_{-1} + \tilde{P}_{-2} - \tilde{P}_{-3} - \tilde{P}_{-4} - \tilde{P}_{-5} + \tilde{P}_{-6} - \epsilon = 0$$

where $\epsilon := 2\sigma_5 - n_{1,5}^0 \geq 0$.

Going on a similar calculation, one gets:

$$\begin{aligned}\epsilon_7 &= 1 + \tilde{P}_{-1} + \tilde{P}_{-2} - \tilde{P}_{-5} - \tilde{P}_{-6} + \tilde{P}_{-7} - 2\sigma_5 + 2n_{1,5}^0 + n_{1,6}^0 \\ \epsilon_8 &= 2\tilde{P}_{-1} + \tilde{P}_{-2} + \tilde{P}_{-3} - \tilde{P}_{-4} - \tilde{P}_{-5} - \tilde{P}_{-7} + \tilde{P}_{-8} \\ &\quad - 3\sigma_5 + 3n_{1,5}^0 + 2n_{1,6}^0 + n_{1,7}^0\end{aligned}$$

2.8. Geometric inequalities. We say that a formal basket $\mathbf{B} = (B, \tilde{P}_{-1})$ is geometric if $\mathbf{B} = (B_X, P_{-1}(X))$ for a terminal weak \mathbb{Q} -Fano 3-fold X . By [7], one has that $-K_X \cdot c_2(X) \geq 0$. Therefore [8, 10.3] gives the following inequality:

$$\gamma(B) := \sum_{i=1}^t \frac{1}{r_i} - \sum_{i=1}^t r_i + 24 \geq 0 \quad (2.1)$$

Furthermore $-K^3(\mathbf{B}) = -K_X^3 > 0$ gives the inequality:

$$\sigma'(B) < 2\tilde{P}_{-1} + \sigma(B) - 6. \quad (2.2)$$

Moreover by [6, Lemma 15.6.2], whenever $P_{-m} > 0$ and $P_{-n} > 0$, one has

$$P_{-m-n} \geq P_{-m} + P_{-n} - 1. \quad (2.3)$$

3. Plurigenus

We begin with the following observation, which follows immediately from the definition of packing and γ :

Lemma 3.1. *Given a packing of baskets $B \succ B'$, we have $\gamma(B) > \gamma(B')$. In particular, if inequality (2.1) doesn't hold for B , then it doesn't hold for B' .*

3.2. Natation and Convention. For simplicity, we write P_{-m} for \tilde{P}_{-m} in what follows.

In this section, we mainly study those formal baskets (B, P_{-1}) satisfying inequalities (2.1) and (2.2).

We may and often do abuse the notation of \mathbf{B} with B when \tilde{P}_{-1} is given.

The following proposition provides an evidence about how our method is going to work effectively.

Proposition 3.3. *Given $p_i \in \mathbb{Z}^+$, there are only finitely many formal baskets admitting $(P_{-1}, P_{-2}) = (p_1, p_2)$ and satisfying (2.1).*

Proof. The number of pairs in B is upper bounded by $\sigma = 10 - 5p_1 + p_2$. Assume $B = \{(b_i, r_i)\}$. Then inequality (2.1) gives

$$\sum_{i=1}^t r_i \leq 24 + \sum_i \frac{1}{r_i} \leq 24 + \frac{\sigma}{2}.$$

Clearly B has finite number of possibilities. This completes the proof. \square

3.4. Geometrically constrained baskets with $P_{-2} = 0$. We now study formal baskets, satisfying (2.1) and (2.2), with $P_{-1} = P_{-2} = 0$ and will give a complete classification in this situation. In fact, some other geometric constrains such as $P_{-2} \geq P_{-1}$ are tacitly employed in our argument.

Given a formal basket $\mathbf{B} = (B, 0)$ with $P_{-1} = P_{-2} = 0$. The initial basket $B^{(0)}(B)$ has datum:

$$\begin{aligned} n_{1,2}^0 &= 5 - P_{-3}, \\ n_{1,3}^0 &= 4 + 3P_{-3} - P_{-4}, \\ n_{1,4}^0 &= 1 - 2P_{-3} + P_{-4} - \sigma_5. \end{aligned}$$

By Lemma 3.1, $B^{(0)}(B)$ satisfies (2.1) and thus

$$\begin{aligned} 0 &\leq \gamma(B^{(0)}(B)) = \sum_{r \geq 2} \left(\frac{1}{r} - r\right) n_{1,r}^0 + 24 \\ &\leq \sum_{r=2,3,4} \left(\frac{1}{r} - r\right) n_{1,r}^0 - \frac{24}{5} \sigma_5 + 24 \\ &= \frac{25}{12} + P_{-3} - \frac{13}{12} P_{-4} - \frac{21}{20} \sigma_5. \end{aligned}$$

It follows that

$$P_{-4} + \sigma_5 \leq P_{-3} + 1. \quad (3.1)$$

We need a more refined inequality, due to the fact that $B^{(5)}$ satisfies (2.1) again by Lemma 3.1. Because $\gamma(B^{(5)}(B)) = \gamma(B^{(0)}(B)) - \frac{19}{30} \epsilon_5$, one gets

$$0 \leq \frac{25}{12} + P_{-3} - \frac{13}{12} P_{-4} - \frac{21}{20} \sigma_5 - \frac{19}{30} \epsilon_5. \quad (3.2)$$

On the other hand, by $n_{1,4}^0 \geq 0$, we have

$$P_{-4} \geq 2P_{-3} - 1.$$

Thus we conclude that $(P_{-3}, P_{-4}) = (0, 0), (0, 1), (1, 1), (1, 2), (2, 3)$.

Here is our complete classification:

Theorem 3.5. *Any geometric basket with $P_{-2} = 0$ is among the following list:*

Table A

B	$-K^3$	P_{-3}	P_{-4}	P_{-5}	P_{-6}	P_{-7}	P_{-8}
No.1. $\{2 \times (1, 2), 3 \times (2, 5), (1, 3), (1, 4)\}$	1/60	0	0	1	1	1	2
No.2. $\{5 \times (1, 2), 2 \times (1, 3), (2, 7), (1, 4)\}$	1/84	0	1	0	1	1	2
No.3. $\{5 \times (1, 2), 2 \times (1, 3), (3, 11)\}$	1/66	0	1	0	1	1	2
No.4. $\{5 \times (1, 2), (1, 3), (3, 10), (1, 4)\}$	1/60	0	1	0	1	1	2
No.5. $\{5 \times (1, 2), (1, 3), 2 \times (2, 7)\}$	1/42	0	1	0	1	2	3
No.6. $\{4 \times (1, 2), (2, 5), 2 \times (1, 3), 2 \times (1, 4)\}$	1/30	0	1	1	2	2	4
No.7. $\{3 \times (1, 2), (2, 5), 5 \times (1, 3)\}$	1/30	1	1	1	3	3	4
No.8. $\{2 \times (1, 2), (3, 7), 5 \times (1, 3)\}$	1/21	1	1	1	3	4	5
No.9. $\{(1, 2), (4, 9), 5 \times (1, 3)\}$	1/18	1	1	1	3	4	5
No.10. $\{3 \times (1, 2), (3, 8), 4 \times (1, 3)\}$	1/24	1	1	1	3	3	5
No.11. $\{3 \times (1, 2), (4, 11), 3 \times (1, 3)\}$	1/22	1	1	1	3	3	5
No.12. $\{3 \times (1, 2), (5, 14), 2 \times (1, 3)\}$	1/21	1	1	1	3	3	5
No.13. $\{2 \times (1, 2), 2 \times (2, 5), 4 \times (1, 3)\}$	1/15	1	1	2	4	5	7
No.14. $\{(1, 2), (3, 7), (2, 5), 4 \times (1, 3)\}$	17/210	1	1	2	4	6	8
No.15. $\{2 \times (1, 2), (2, 5), (3, 8), 3 \times (1, 3)\}$	3/40	1	1	2	4	5	8
No.16. $\{2 \times (1, 2), (5, 13), 3 \times (1, 3)\}$	1/13	1	1	2	4	5	8
No.17. $\{(1, 2), 3 \times (2, 5), 3 \times (1, 3)\}$	1/10	1	1	3	5	7	10
No.18. $\{4 \times (1, 2), 5 \times (1, 3), (1, 4)\}$	1/12	1	2	2	5	6	9
No.19. $\{4 \times (1, 2), 4 \times (1, 3), (2, 7)\}$	2/21	1	2	2	5	7	10
No.20. $\{4 \times (1, 2), 3 \times (1, 3), (3, 10)\}$	1/10	1	2	2	5	7	10
No.21. $\{3 \times (1, 2), (2, 5), 4 \times (1, 3), (1, 4)\}$	7/60	1	2	3	6	8	12
No.22. $\{3 \times (1, 2), 7 \times (1, 3)\}$	1/6	2	3	4	9	12	17
No.23. $\{2 \times (1, 2), (2, 5), 6 \times (1, 3)\}$	1/5	2	3	5	10	14	20

Proof. This theorem follows from Propositions 3.6, 3.7. \square

Proposition 3.6. *If $(P_{-3}, P_{-4}) = (0, 0)$, then B is of type No.1 in Table A.*

Proof. Now $\sigma_5 \leq 1$ and $\epsilon_5 = 2 + P_{-5} - \sigma_5 \leq 3$ by (3.2).

Claim 1. The situation $(\sigma_5, P_{-5}) = (0, 0)$ does not happen.

Proof of the claim. We have

$$B^{(5)}(B) = \{3 \times (1, 2), 2 \times (2, 5), 2 \times (1, 3), (1, 4)\}.$$

If $B = B^{(5)}(B)$, then $-K^3(B) = \sigma - \sigma' - 6 = -\frac{1}{60} < 0$, a contradiction. Thus $B \neq B^{(5)}(B)$. Assume that B has totally t pairs. Then $t < 8$ since $B^{(5)} \succ B$. From $B^{(5)}(B)$ we know $\sum_i r_i = 26$. Thus (2.1) becomes $\sum_{i=1}^t \frac{1}{r_i} \geq 2$. Assume $r_1 \leq r_2 \leq \dots \leq r_t$. If $t \leq 4$, then $r_t > 2$ and $\sum \frac{1}{r_i} \leq \frac{3}{2} + \frac{1}{r_t} < 2$. So (2.1) fails. If $t = 5$, we consider the value of r_3 . Whenever $r_3 = 2$, one has $r_4 \geq 3$ and $26 = 6 + r_4 + r_5 \leq 6 + 2r_5$ gives $r_5 \geq 10$. Thus $\sum_i \frac{1}{r_i} \leq \frac{3}{2} + \frac{1}{3} + \frac{1}{10} < 2$. So (2.1) fails. Whenever $r_3 \geq 3$, then $r_4 \geq 3$ and $r_5 \geq 7$. So again $\sum_i \frac{1}{r_i} \leq 1 + \frac{2}{3} + \frac{1}{7} < 2$, a contradiction to (2.1). Therefore we have seen $t = 6, 7$, which means that B is exactly obtained by 1 or 2 prime packings from $B^{(5)}(B)$.

When $t = 7$, B must be one of the following cases:

- (I). $\{2 \times (1, 2), (3, 7), (2, 5), 2 \times (1, 3), (1, 4)\}$; $-K^3 = -\frac{1}{60} < 0$ (contradiction);
- (II). $\{3 \times (1, 2), (2, 5), (3, 8), (1, 3), (1, 4)\}$; $-K^3 = -\frac{1}{120} < 0$ (contradiction);
- (III). $\{3 \times (1, 2), 2 \times (2, 5), (1, 3), (2, 7)\}$; $-K^3 = -\frac{1}{210} < 0$ (contradiction);

When $t = 6$, B is nothing but an extra prime packing from one of I, II and III:

- (I-1). $\{(1, 2), 2 \times (3, 7), 2 \times (1, 3), (1, 4)\}$; $\sum_i \frac{1}{r_i} < 2$ (contradiction);
- (I-2). $\{(1, 2), (4, 9), (2, 5), 2 \times (1, 3), (1, 4)\}$; $\sum_i \frac{1}{r_i} < 2$ (contradiction);
- (I-3). $\{2 \times (1, 2), (5, 12), 2 \times (1, 3), (1, 4)\}$; $-K^3 = 0$ (contradiction);
- (I-4). $\{2 \times (1, 2), (3, 7), (3, 8), (1, 3), (1, 4)\}$; $\sum_i \frac{1}{r_i} < 2$ (contradiction);
- (I-5). $\{2 \times (1, 2), (3, 7), (2, 5), (1, 3), (2, 7)\}$; $\sum_i \frac{1}{r_i} < 2$ (contradiction);
- (II-1). $\{3 \times (1, 2), (5, 13), (1, 3), (1, 4)\}$; $-K^3 = -\frac{1}{156} < 0$ (contradiction);
- (II-2). $\{3 \times (1, 2), (2, 5), (4, 11), (1, 4)\}$; $-K^3 = -\frac{1}{220} < 0$ (contradiction);
- (II-3). $\{3 \times (1, 2), (2, 5), (3, 8), (2, 7)\}$; $\sum_i \frac{1}{r_i} < 2$ (contradiction);
- (III-1). $\{3 \times (1, 2), 2 \times (2, 5), (3, 10)\}$; $-K^3 = 0$ (contradiction);

□

We go on proving Proposition 3.6.

If $\sigma_5 = 0$ and $P_{-5} > 0$. Because $2 + P_{-5} = \epsilon_5 \leq 3$, we see $P_{-5} = 1$ and $B^{(5)}(B) = \{2 \times (1, 2), 3 \times (2, 5), (1, 3), (1, 4)\}$. A computation shows

that any non-trivial packing of $B^{(5)}$ has $\gamma < 0$. Hence $B = B^{(5)}(B)$. So B corresponds to case No.1 in Table A.

If $\sigma_5 = 1$ and $P_{-5} = 0$, then we have $B^{(5)}(B) = \{4 \times (1, 2), (2, 5), 3 \times (1, 3), (1, s)\}$ with $s \geq 5$, $-K^3 = \frac{1}{5} - \frac{1}{s}$ and $\gamma = 5 - s + \frac{1}{5} + \frac{1}{s}$. When $s \geq 6$, we have $\gamma < 0$, a contradiction. Hence we must have $s = 5$. Since $-K^3(B^{(5)}) = 0$, so $B^{(5)}(B) \succ B$ is nontrivial. However, any non-trivial packing of $B^{(5)}(B)$ has $\gamma < 0$, which still gives a contradiction. Thus this case can not happen.

Finally if $\sigma_5 = 1$ and $P_{-5} > 0$, then we get a contradiction from (3.2).

We have proved Proposition 3.6. \square

Proposition 3.7. (1) If $(P_{-3}, P_{-4}) = (0, 1)$, then B is of type No.2-No.6 in Table A;

(2) If $(P_{-3}, P_{-4}) = (1, 1)$, then B is of type No.7-No.17 in Table A;

(3) If $(P_{-3}, P_{-4}) = (1, 2)$, then B is of type No.18-No.21 in Table A;

(4) If $(P_{-3}, P_{-4}) = (2, 3)$, then B is of type No. 22, No. 23 in Table A.

Proof. (1) By (3.1), we have $\sigma_5 = 0$, hence $B^{(0)}(B) = \{5 \times (1, 2), 3 \times (1, 3), 2 \times (1, 4)\}$. By (3.2), we have $P_{-5} = \epsilon_5 \leq 1$.

If $P_{-5} = 0$, then we can easily compute all possible formal baskets B with $B^{(5)}(B) = \{5 \times (1, 2), 3 \times (1, 3), 2 \times (1, 4)\}$, $\gamma > 0$ and $-K^3(B) > 0$. In fact, by classifying all baskets with $B^{(5)}(B)$ as above, and verifying inequalities (2.1), (2.2), the reader should have no difficulty to see that B has 4 types which correspond to No.2 through No.5 in Table A.

If $P_{-5} = 1$, then $B^{(5)}(B) = \{4 \times (1, 2), (2, 5), 2 \times (1, 3), 2 \times (1, 4)\}$. Because any basket dominated by $B^{(5)}(B)$ has $\gamma < 0$, we see $B = B^{(5)}(B)$ which corresponds to No. 6 in Table A.

(2) In this case, $0 \leq n_{1,4}^0 = -\sigma_5$ gives $\sigma_5 = 0$. By (3.2), we have $\epsilon_5 \leq 3$.

If $\epsilon_5 = 0$, then we get $B = B^{(5)}(B) = \{4 \times (1, 2), 6 \times (1, 3)\}$ with $-K^3(B) = 0$, a contradiction.

If $\epsilon_5 = 1$, then we get $B^{(5)}(B) = \{3 \times (1, 2), (2, 5), 5 \times (1, 3)\}$. By computation, we see that B corresponds to No. 7 through No. 12 in Table A.

If $\epsilon_5 = 2$, then we get $B^{(5)}(B) = \{2 \times (1, 2), 2 \times (2, 5), 4 \times (1, 3)\}$. We see that B corresponds to No. 13 through No. 16 in Table A.

If $\epsilon_5 = 3$, then we get $B^{(5)}(B) = \{(1, 2), 3 \times (2, 5), 3 \times (1, 3)\}$. We see that B has only one possibility, which is $B^{(5)}(B)$ corresponding to No. 17 in Table A.

(3) By (3.1), we must have $\sigma_5 = 0$. Moreover, by (3.2), we have $\epsilon_5 \leq 1$.

If $\epsilon_5 = 0$, then we get $B^{(5)}(B) = \{4 \times (1, 2), 5 \times (1, 3), (1, 4)\}$ and B corresponds to No. 18 through No. 20 in Table A.

If $\epsilon_5 = 1$, then we get $B^{(5)}(B) = \{3 \times (1, 2), (2, 5), 4 \times (1, 3), (1, 4)\}$ and $B = B^{(5)}(B)$ corresponds to No. 21 in Table A.

(4) By (3.1), we must have $\sigma_5 = 0$. Moreover, by (3.2), we have $\epsilon_5 \leq 1$.

If $\epsilon_5 = 0$, then we get $B^{(5)}(B) = \{3 \times (1, 2), 7 \times (1, 3)\}$ and $B = B^{(5)}(B)$ corresponds to No.22 in Table A.

If $\epsilon_5 = 1$, then we get $B^{(5)}(B) = \{2 \times (1, 2), (2, 5), 6 \times (1, 3)\}$ and $B = B^{(5)}(B)$ corresponds to No.23 in Table A. \square

Now we are able to prove the following

Theorem 3.8. *Let X be a terminal weak \mathbb{Q} -Fano 3-fold. Then $P_{-6} > 0$.*

Proof. Set $B := B_X$. If $P_{-6} = 0$, then $P_{-1} = P_{-2} = P_{-3} = 0$. By $\epsilon_6 = 0$, we get $P_{-4} = P_{-5} = \epsilon = 0$. Thus $B^{(5)}(B) = \{3 \times (1, 2), 2 \times (2, 5), 2 \times (1, 3), (1, 4)\}$. By Claim 1 in the proof of Proposition 3.6, we know that such a basket B does not exist. Thus $P_{-6}(X) > 0$. \square

Proposition 3.9. *Let X be a terminal weak \mathbb{Q} -Fano 3-fold. Then $P_{-4} > 0$ unless $B_X = \{2 \times (1, 2), 3 \times (2, 5), (1, 3), (1, 4)\}$.*

Proof. Assume $P_{-4} = 0$. Then clearly $P_{-1} = P_{-2} = 0$. Recall that $n_{1,4}^0 = 1 - 2P_{-3} + P_{-4} - \sigma_5$. It follows that $P_{-3} = 0$. By Proposition 3.6, we see $B = \{2 \times (1, 2), 3 \times (2, 5), (1, 3), (1, 4)\}$. \square

Proposition 3.10. *Let X be a terminal weak \mathbb{Q} -Fano 3-fold. If $P_{-2} > 0$, then $P_{-2k} \geq 2$ for all $k \geq 4$.*

Proof. If $P_{-2} \geq 2$, then there is nothing to prove. Thus it remains to consider the case $P_{-2} = 1$. (Actually we will prove that $P_{-6} \geq 2$ except for a very special case.)

Case 1. $P_{-1} = 0$.

Then $n_{1,4}^0 = -2P_{-3} + P_{-4} - \sigma_5 \geq 0$. Note that $P_{-4} \geq 2$ whenever $P_{-3} > 0$. We only need to consider the case $P_{-3} = 0$ and $P_{-4} = 1$. Since $n_{1,4}^0 = 1 - \sigma_5$, we see $\sigma_5 \leq 1$.

If $\sigma_5 = 0$, then $\epsilon = 0$ and $\epsilon_5 = 1 + P_{-5} \leq n_{1,3}^0 = 1$. Thus $P_{-5} = 0$. Now $\epsilon_6 = 0$ gives

$$2 \leq P_{-2} + P_{-6} = P_{-3} + P_{-4} + P_{-5} + \epsilon = 1,$$

a contradiction. Thus $\sigma_5 = 1$.

Now if $P_{-5} > 0$, then $P_{-3} + P_{-4} + P_{-5} + \epsilon \geq 3$ and thus $\epsilon_6 = 0$ gives $P_{-6} \geq 2$. Clearly $P_{-8} \geq 2$.

If $P_{-5} = 0$, then $\epsilon_5 = 0$ and $B^{(5)}(B) = \{9 \times (1, 2), (1, 3), (1, s)\}$ with $s \geq 5$. By our definition, $B^{(n)}(B) = B^{(5)}(B)$ for all $n \geq 5$. Also notice that $B = B^{(n)}(B)$ for n sufficiently large. We thus have $B = B^{(5)}(B)$.

When $\epsilon = 1$, then $n_{1,5}^0 = 1$ and $n_{1,r}^0 = 0$ for all $r \geq 6$, which means $s = 5$. Now $\sigma'(B) = \frac{9}{2} + \frac{1}{3} + \frac{1}{5} > 5$ and (2.2) fails. Thus we have $\epsilon \geq 2$. Hence $\epsilon_6 = 0$ implies $P_{-6} = \epsilon \geq 2$.

Case 2. $P_{-1} = 1$.

We may assume $P_{-6} = 1$. Then $P_{-2} = P_{-3} = P_{-4} = P_{-5} = 1$. Since $\epsilon_6 = 0$, one gets $\epsilon = 2$ and therefore $\sigma_5 > 0$. Note that $\epsilon_5 = 2 - \sigma_5 \geq 0$. We have $\sigma_5 \leq 2$.

If $\sigma_5 = 2$, then $n_{1,5}^0 = 2$. We have $B^{(5)}(B) = \{2 \times (1, 2), 2 \times (1, 3), 2 \times (1, 5)\}$. By the same reason as above, $B = B^{(5)}(B)$. Because $\sigma'(B) = 1 + \frac{2}{3} + \frac{2}{5} > 2$, so (2.2) fails.

Thus $\sigma_5 = 1$, and then we have $B^{(5)}(B) = \{(1, 2), (2, 5), (1, 3), (1, 4), (1, s)\}$ with $s \geq 6$. If $s \geq 8$, then $\epsilon_8 \geq 0$ gives

$$P_{-8} - P_{-7} = \epsilon_8 + 1 \geq 1.$$

Since $P_{-7} \geq P_{-6} \geq 1$, we have $P_{-8} \geq 2$.

We now assume that $s = 6, 7$. Considering all baskets with given $B^{(5)}$, we may find that they dominate one of the following minimal elements:

$$B_1 = \{(3, 7), (2, 7), (1, s)\};$$

$$B_2 = \{(1, 2), (3, 8), (1, 4), (1, s)\}.$$

Because $\sigma'(B) \geq \sigma'(B_i) \geq 2$ whenever $s = 6, 7$ and $i = 1, 2$, inequality (2.2) fails for all B , which says that this case does not happen.

We have actually proved $P_{-8} \geq 2$. Furthermore $P_{-6} \geq 2$ except when $P_{-1} = 1$ and $\sigma_5 = 1$.

This completes the proof. \square

Now we prove the following:

Theorem 3.11. *Let X be a terminal weak \mathbb{Q} -Fano 3-fold. Then $P_{-2k} \geq 2$ for all $k \geq 4$.*

Proof. When $P_{-2} > 0$, then this follows from Proposition 3.10.

When $P_{-2} = 0$, then it follows from Theorem 3.5 and by computing P_{-2k} for each case in the list. \square

4. The anti-volume

By Riemann-Roch formula directly, we have

$$\begin{aligned} \frac{1}{2}(-K^3) &= P_{-1} - 3 + l(2), \\ \frac{5}{2}(-K^3) &= P_{-2} - 5 + l(3). \end{aligned}$$

4.1. An inequality. We have $B^{(0)}(B) \succ B$ and so $(B^{(0)}(B), P_{-1}) \succ (B, P_{-1})$. By our formulae in Section 2, we get

$$\sigma'(B^{(0)}) - K^3(B^{(0)}) = \sigma'(B) - K^3(B) = 2P_{-1} + \sigma(B) - 6.$$

We have

$$\begin{aligned}\sigma'(B^{(0)}(B)) &= \frac{1}{2}n_{1,2}^0 + \frac{1}{3}n_{1,3}^0 + \frac{1}{4}n_{1,4}^0 + \sum_{\sigma_5} \frac{1}{r}n_{1,r}^0 \\ &\leq \frac{1}{2}n_{1,2}^0 + \frac{1}{3}n_{1,3}^0 + \frac{1}{4}(n_{1,4}^0 + \sigma_5) \\ &= \frac{1}{12}(49 - 35P_{-1} + 13P_{-2} - P_{-4}).\end{aligned}$$

Thus we get the following inequality by 2.4(iv):

$$\begin{aligned}-K^3(B) &\geq -K^3(B^{(0)}(B)) = 2P_{-1} + \sigma(B) - 6 - \sigma'(B^{(0)}(B)) \\ &\geq \frac{1}{12}(-1 - P_{-1} - P_{-2} + P_{-4}).\end{aligned}\tag{4.1}$$

In particular, we have $-K^3 \geq \frac{1}{12}$ whenever $P_{-4} > P_{-2} + P_{-1} + 1$.

Lemma 4.2. *Assume $P_{-4} = P_{-2} + P_{-1} + 1$. Then:*

- (1). $-K^3 \geq \frac{1}{20}$ when $\sigma_5 > 0$;
- (2). $-K^3 \geq \frac{1}{30}$ when $\epsilon_5 > 0$.

Proof. If $\sigma_5 > 0$, then our computation in 4.1 gives:

$$-K^3(B) \geq -K^3(B^{(0)}(B)) \geq \frac{1}{4}\sigma_5 - \sum_{\sigma_5}^{\geq 5} \frac{1}{r} \geq \frac{1}{20}.$$

If $\epsilon_5 > 0$ and $\sigma_5 = 0$, then

$$\begin{aligned}\sigma'(B^{(5)}(B)) &= \sigma'(B^{(0)}(B)) - \frac{1}{30}\epsilon_5 \\ &\leq \sigma'(B^{(0)}(B)) - \frac{1}{30}.\end{aligned}$$

Therefore $-K^3(B) \geq -K^3(B^{(5)}(B)) \geq -K^3(B^{(0)}(B)) + \frac{1}{30} \geq \frac{1}{30}$. \square

4.3. Assumption. Under the situation $P_{-4} = P_{-2} + P_{-1} + 1$, we only need to study the case $\sigma_5 = \epsilon_5 = 0$.

Now we are prepared to prove the following:

Theorem 4.4. *Let X be a terminal weak \mathbb{Q} -Fano 3-fold. Then*

$$-K^3(X) \geq \frac{1}{330}.$$

Proof. By (4.1) and Lemma 4.2, we only need to study one of the following situations:

- (i) $P_{-4} < P_{-2} + P_{-1} + 1$;
- (ii) $P_{-4} = P_{-2} + P_{-1} + 1$, $\sigma_5 = 0$ and $\epsilon_5 = 0$.

Case I. $P_{-1} = 0$. We have $\sigma = 10 + P_{-2} \geq 10$.

Subcase I-1. $P_{-2} \geq 3$. Then $P_{-4} \geq 2P_{-2} - 1 > P_{-2} + P_{-1} + 1$. By (4.1), we have $-K^3 \geq \frac{1}{12}$.

Subcase I-2. $P_{-2} = 2$. Then $n_{1,3}^0 \geq 0$ and $n_{1,4}^0 \geq 0$ gives

$$3P_{-3} \geq P_{-4} \geq 1 + 2P_{-3},$$

which implies that $P_{-3} \geq 1$.

If $P_{-3} \geq 2$, then $P_{-4} \geq 5 > P_{-2} + P_{-1} + 1$. By (4.1), we see $-K^3(B) \geq \frac{1}{12}$.

If $P_{-3} = 1$, then $P_{-4} = 3$. We have $B^{(0)}(B) = \{12 \times (1, 2)\}$. Clearly $B^{(0)}(B)$ admits no packing. So $B = B^{(0)}(B)$. However $-K^3(B) = 0$, a contradiction.

Subcase I-3. $P_{-2} = 1$. By $n_{1,4}^0 \geq 0$ and $n_{1,3}^0 \geq 0$, we get

$$2 + 3P_{-3} \geq P_{-4} \geq 2P_{-3}.$$

Also, if $P_{-4} \geq 3$, then (4.1) gives $-K^3(B) \geq \frac{1}{12}$. Thus we only need to consider the situations: $(P_{-3}, P_{-4}) = (0, 1), (0, 2), (1, 2)$.

If $(P_{-3}, P_{-4}) = (1, 2)$, then $B^{(0)}(B) = \{8 \times (1, 2), 3 \times (1, 3)\}$ with $-K^3(B^{(0)}) = 0$. Thus B must be a packing of $B^{(0)}$. The one-step packing $B_1 = \{7 \times (1, 2), (2, 5), 2 \times (1, 3)\}$ has $-K^3(B_1) = \frac{1}{30} > 0$. Because $B_1 \succ B$, we see $-K^3(B) \geq \frac{1}{30}$.

If $(P_{-3}, P_{-4}) = (0, 2)$, then $P_{-4} = P_{-2} + P_{-1} + 1$. By our assumption, we may assume $\sigma_5 = \epsilon_5 = 0$. So $B^{(0)}(B) = \{9 \times (1, 2), 2 \times (1, 4)\}$. Since $B^{(0)}$ admits no prime packing, $B = B^{(0)}(B)$ and $-K^3(B) = 0$, a contradiction.

Finally we consider the situation $(P_{-3}, P_{-4}) = (0, 1)$. $n_{1,4}^0 \geq 0$ gives $\sigma_5 \leq 1$. If $\sigma_5 = 0$, we have $B^{(0)}(B) = \{9 \times (1, 2), (1, 3), (1, 4)\}$ with $-K^3(B^{(0)}) = -\frac{1}{12} < 0$. Considering a minimal basket B_{min} dominated by $B^{(0)}(B)$, then either $B_{min} = \{(10, 21), (1, 4)\}$ with $-K^3(B_{min}) = -\frac{1}{84} < 0$ or $B_{min} = \{9 \times (1, 2), (2, 7)\}$ with $-K^3 = -\frac{1}{14} < 0$. Thus we see $-K^3(B) \leq -K^3(B_{min}) < 0$, a contradiction.

If $\sigma_5 = 1$, we see $B^{(0)}(B) = \{9 \times (1, 2), (1, 3), (1, s)\}$ with $s \geq 5$ and $-K^3(B^{(0)}(B)) = \frac{1}{6} - \frac{1}{s}$. Notice that baskets dominated by $B^{(0)}(B)$ are linearly ordered.

For $s \geq 7$, we have $-K^3(B) \geq -K^3(B^{(0)}(B)) \geq \frac{1}{42}$. For $s = 6$, $-K^3(B^{(0)}) = 0$ and the one step packing is $B_1 = \{8 \times (1, 2), (2, 5), (1, 6)\}$ with $-K^3 = \frac{1}{30}$. Thus $-K^3(B) \geq -K^3(B_1) \geq \frac{1}{30}$. For the last case $s = 5$, B must be dominated by $B_2 = \{7 \times (1, 2), (3, 7), (1, 5)\}$ with $-K^3(B_2) = \frac{1}{70}$. We see $-K^3(B) \geq -K^3(B_2) \geq \frac{1}{70}$.

Subcase I-4. $P_{-2} = 0$. By Proposition 3.5, we know $-K^3(B) \geq \frac{1}{84}$.

This completes the proof for Case I.

Case II. $P_{-1} = 1$. We have $\sigma = 5 + P_{-2} \geq 5$.

Subcase II-1. $P_{-2} \geq 4$. Then $P_{-4} \geq 2P_{-2} - 1 \geq P_{-2} + 3 > P_{-2} + P_{-1} + 1$ by (2.3). According to (4.1), we have $-K^3(B) \geq \frac{1}{12}$.

Subcase II-2. $P_{-2} = 3$. Then $P_{-4} \geq 5 = P_{-2} + P_{-1} + 1$. By our assumption, we only need to consider the situation $P_{-4} = 5$ and $\sigma_5 =$

$\epsilon_5 = 0$. Now $n_{1,4}^0 \geq 0$ gives $P_{-3} = 3$ and thus $B^{(0)}(B) = \{8 \times (1, 2)\}$ with $-K^3(B^{(0)}(B)) = 0$. Since $B^{(0)}(B)$ is already minimal, $B = B^{(0)}(B)$ and thus $-K^3(B) = 0$, a contradiction.

Subcase II-3. $P_{-2} = 2$. Notice that $P_{-3} \geq P_{-2} = 2$ and $P_{-4} \geq 2P_{-2} - 1 = 3$. In fact, if $P_{-4} \geq 5 > P_{-2} + P_{-1} + 1$, we have $-K^3(B) \geq \frac{1}{12}$. From $n_{1,3}^0 \geq 0$ and $n_{1,4}^0 \geq 0$, we get $3P_{-3} - 2 \geq P_{-4} \geq 2P_{-3} - 2$. So it suffices to consider the following situations: $(P_{-3}, P_{-4}) = (2, 3), (2, 4), (3, 4)$.

If $(P_{-3}, P_{-4}) = (3, 4)$, then $B^{(0)}(B) = \{4 \times (1, 2), 3 \times (1, 3)\}$ with $-K^3 = 0$. Consider the one step packing B_1 of $B^{(0)}$, one sees $B_1 = \{3 \times (1, 2), (2, 5), 2 \times (1, 3)\}$ with $-K^3(B_1) = \frac{1}{30} > 0$. Thus $-K^3(B) \geq -K^3(B_1) \geq \frac{1}{30}$.

If $(P_{-3}, P_{-4}) = (2, 4)$, then we may assume that $\sigma_5 = 0$ since $P_{-4} = P_{-2} + P_{-1} + 1$. Thus $B^{(0)}(B) = \{5 \times (1, 2), 2 \times (1, 4)\}$ with $-K^3(B^{(0)}(B)) = 0$. Because $B^{(0)}(B)$ is minimal, $B = B^{(0)}(B)$, a contradiction.

If $(P_{-3}, P_{-4}) = (2, 3)$, then $n_{1,4}^0 \geq 0$ gives $\sigma_5 \leq 1$. Thus either $B^{(0)}(B) = \{5 \times (1, 2), (1, 3), (1, 4)\}$ with $-K^3(B^{(0)}) < 0$ or $B^{(0)} = \{5 \times (1, 2), (1, 3), (1, s)\}$ with $s \geq 5$.

We consider the first case. One can check that any minimal basket dominated by $B^{(0)}$ has negative anti-volume. Thus this case can not happen at all.

Now we consider the later case. If $s \geq 7$, then

$$-K^3(B) \geq -K^3(B^{(0)}(B)) \geq \frac{1}{42}.$$

If $s = 6$, then the one-step packing $B_1 = \{4 \times (1, 2), (2, 5), (1, 6)\}$ with $-K^3(B) \geq -K^3(B_1) = \frac{1}{30}$. If $s = 5$, then the two-step packing $B_2 = \{3 \times (1, 2), (3, 7), (1, 5)\}$ with $-K^3(B) \geq -K^3(B_2) = \frac{1}{70}$.

Subcase II-4. $P_{-2} = 1$. We get $\sigma = 6$ and $-K^3 + \sigma' = 2$. For a similar reason, we only need to consider the situation $P_{-4} \leq 3$. So it remains to consider the cases:

$$(P_{-3}, P_{-4}) = (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)$$

since $P_{-4} \geq P_{-3}$ by $n_{1,4}^0 \geq 0$.

II-4a. If $(P_{-3}, P_{-4}) = (3, 3)$, then $B^{(0)}(B) = \{6 \times (1, 3)\}$ with $-K^3(B^{(0)}) = 0$. No further packing is allowed, so $B = B^{(0)}(B)$, a contradiction.

II-4b. If $(P_{-3}, P_{-4}) = (2, 3)$, by our assumption, we may assume $\sigma_5 = 0$ and $B^{(0)}(B) = \{(1, 2), 3 \times (1, 3), 2 \times (1, 4)\}$ with $-K^3 = 0$. The one-step packing B_1 is either $\{(2, 5), 2 \times (1, 3), 2 \times (1, 4)\}$ with $-K^3(B_1) = \frac{1}{30}$ or $\{(1, 2), 2 \times (1, 3), (2, 7), (1, 4)\}$ with $-K^3(B_1) = \frac{1}{84}$. Thus we see $-K^3(B) \geq \frac{1}{84}$.

II-4c. If $(P_{-3}, P_{-4}) = (1, 3)$, by our assumption, we may assume $\sigma_5 = 0$ and thus $B^{(0)}(B) = \{2 \times (1, 2), 4 \times (1, 4)\}$ with $-K^3 = 0$. This allows no further packings and so $B = B^{(0)}(B)$, a contradiction.

II-4d. If $(P_{-3}, P_{-4}) = (2, 2)$, then $\sigma_5 \leq 1$ by $n_{1,4}^0 \geq 0$. So either $B^{(0)}(B) = \{(1, 2), 4 \times (1, 3), (1, 4)\}$ or $B^{(0)}(B) = \{(1, 2), 4 \times (1, 3), (1, s)\}$ with $s \geq 5$.

In the first situation, every packing of $B^{(0)}(B)$ has negative $-K^3$, which is absurd. Actually, it suffices to check that one minimal basket $\{(5, 14), (1, 4)\}$ has $-K^3 = \frac{-1}{28}$ and that the other minimal basket $\{(1, 2), (5, 16)\}$ has $-K^3 = -\frac{1}{16}$.

In the last situation, we consider the value of s . If $s \geq 7$, then $-K^3(B) \geq -K^3(B^{(0)}(B)) \geq \frac{1}{42}$. If $s = 6$, then the one-step packing is $\{(2, 5), 3 \times (1, 3), (1, 6)\}$ with $-K^3(B) \geq \frac{1}{30}$. If $s = 5$, the one-step packing has $-K^3 = 0$, but the two-step packing is $\{(3, 8), 2 \times (1, 3), (1, 5)\}$ with $-K^3 = \frac{1}{120}$. Hence any further packing gives $-K^3(B) \geq \frac{1}{120}$.

II-4e. If $(P_{-3}, P_{-4}) = (1, 2)$, then $n_{1,4}^0 \geq 0$ gives $\sigma_5 \leq 3$.

If $\sigma_5 \geq 2$, then $\sigma'(B^{(0)}) \leq 1 + \frac{1}{3} + \frac{1}{4} + \frac{2}{5} < 2$ and thus $-K^3(B) \geq -K^3(B^{(0)}(B)) \geq \frac{1}{60}$.

If $\sigma_5 = 0$, then $B^{(0)}(B) = \{2 \times (1, 2), (1, 3), 3 \times (1, 4)\}$ with $-K^3(B^{(0)}(B)) < 0$. Because the only two minimal elements of $B^{(0)}(B)$ are $\{(3, 7), 3 \times (1, 4)\}$ and $\{2 \times (1, 2), (4, 15)\}$ with both negative $-K^3$, so this case does not happen at all.

If $\sigma_5 = 1$, we have $B^{(0)}(B) = \{2 \times (1, 2), (1, 3), 2 \times (1, 4), (1, s)\}$ with $s \geq 5$. When $s \geq 7$, then $-K^3(B) \geq -K^3(B^{(0)}) \geq \frac{1}{42}$. When $s = 6$, then $-K^3(B^{(0)}(B)) = 0$, but the one-step packing of $B^{(0)}(B)$ is either $\{(1, 2), (2, 5), 2 \times (1, 4), (1, 6)\}$ with $-K^3(B) \geq \frac{1}{30}$ or

$$\{2 \times (1, 2), (2, 7), (1, 4), (1, 6)\}$$

with $-K^3(B) \geq \frac{1}{84}$. When $s = 5$, then we have

$$B^{(0)}(B) = \{2 \times (1, 2), (1, 3), 2 \times (1, 4), (1, 5)\}$$

with negative $-K^3$. By computing all possible packings, one can find out that B can be obtained by packing either $\{(3, 7), 2 \times (1, 4), (1, 5)\}$ with $-K^3 = \frac{1}{70}$ or $\{(1, 2), (2, 5), (1, 4), (2, 9)\}$ with $-K^3 = \frac{1}{180}$. Thus we have proved $-K^3(B) \geq \frac{1}{180}$.

II-4f. If $(P_{-3}, P_{-4}) = (1, 1)$, then $n_{1,4}^0 \geq 0$ gives $\sigma_5 \leq 2$.

If $\sigma_5 = 0$, then $B^{(0)} = \{2 \times (1, 2), 2 \times (1, 3), 2 \times (1, 4)\}$. By calculation, one sees that all minimal elements dominated by $B^{(0)}$ have negative $-K^3$. Thus this case doesn't happen.

If $\sigma_5 = 1$, then $B^{(0)} = \{2 \times (1, 2), 2 \times (1, 3), (1, 4), (1, s)\}$ with $s \geq 5$. When $s = 5$, each minimal element dominated by $B^{(0)}$ has negative $-K^3$. In fact, they are $\{2 \times (2, 5), (2, 9)\}$ and $\{2 \times (1, 2), (3, 10), (1, 5)\}$. Thus this case doesn't happen.

When $s \geq 6$, by calculation, we see that B is dominated by one of the following baskets and thus $-K^3(B)$ has the lower bounds accordingly:

- $\{2 \times (1, 2), 2 \times (1, 3), (1, 4), (1, s)\}$ with $s \geq 13$ and $-K^3 = \frac{1}{12} - \frac{1}{s} \geq \frac{1}{156}$. But when $s \geq 13$, $\gamma < 0$. So this case can not happen;
- $\{2 \times (1, 2), (1, 3), (2, 7), (1, s)\}$ with $s = 11, 12$ and $-K^3 = \frac{2}{21} - \frac{1}{s} \geq \frac{1}{231}$;
- $\{(1, 2), (2, 5), (1, 3), (1, 4), (1, s)\}$ with $s = 9, 10, 11, 12$ and $-K^3 = \frac{7}{60} - \frac{1}{s} \geq \frac{1}{180}$;
- $\{(3, 7), (1, 3), (1, 4), (1, 8)\}$ with $-K^3 = \frac{1}{168}$;
- $\{(1, 2), (2, 5), (2, 7), (1, 8)\}$ with $-K^3 = \frac{1}{280}$;
- $\{2 \times (2, 5), (1, 4), (1, s)\}$ with $s = 7, 8, 9, 10, 11, 12$ and $-K^3 = \frac{3}{20} - \frac{1}{s} \geq \frac{1}{140}$.

Finally if $\sigma_5 = 2$, $B^{(0)}(B) = \{2 \times (1, 2), 2 \times (1, 3), (1, s_1), (1, s_2)\}$ with $s_2 \geq s_1 \geq 5$. First when $\frac{1}{s_1} + \frac{1}{s_2} < \frac{1}{3}$, $-K^3(B^{(0)}) > 0$. In particular, if $s_1 + s_2 \geq 13$, we see $-K^3(B) \geq -K^3(B^{(0)}) \geq \frac{1}{120}$. We are left to consider the situations: $(s_1, s_2) = (6, 6), (5, 7), (5, 6)$ and $(5, 5)$.

When $(s_1, s_2) = (6, 6)$, we see that B is dominated by

$$B_6 = \{(1, 2), (2, 5), (1, 3), 2 \times (1, 6)\}$$

and thus $-K^3(B) \geq -K^3(B_6) = \frac{1}{30}$.

When $(s_1, s_2) = (5, 7)$, we see that B is dominated by

$$B_7 = \{(1, 2), (2, 5), (1, 3), (1, 5), (1, 7)\}$$

with $-K^3(B) \geq -K^3(B_7) = \frac{1}{42}$.

When $(s_1, s_2) = (5, 6)$, we see that B is dominated by one of the following baskets B_8 with $-K^3(B) \geq -K^3(B_8)$:

- $\{2 \times (2, 5), (1, 5), (1, 6)\}$ with $-K^3 = \frac{1}{30}$;
- $\{(3, 7), (1, 3), (1, 5), (1, 6)\}$ with $-K^3 = \frac{1}{70}$;
- $\{(1, 2), (3, 8), (1, 5), (1, 6)\}$ with $-K^3 = \frac{1}{120}$;
- $\{(1, 2), (2, 5), (1, 3), (2, 11)\}$ with $-K^3 = \frac{1}{330}$.

When $(s_1, s_2) = (5, 5)$, because any minimal element dominated by $B^{(0)}(B)$ has negative $-K^3$, we see that this case doesn't happen at all.

This completes the proof for Case II.

Case III. $P_{-1} = 2$. We have $\sigma = P_{-2} \geq 2P_{-1} - 1 = 3$.

Subcase III-1. $P_{-2} \geq 5$. Since $\sigma = 5$, one gets

$$\begin{aligned} l(3) &= \sum_i \left\{ \frac{b_i(r_i - b_i)}{2r_i} + \frac{b_i(r_i - 2b_i)}{r_i} \right\} \\ &\geq \sum_i \frac{b_i(r_i - b_i)}{2r_i} \geq \frac{5}{4}. \end{aligned}$$

So $-K^3 \geq \frac{1}{2}$ by Riemann-Roch formula directly.

Subcase III-2. $P_{-2} = 4$. If $B^{(0)}(B) = \{4 \times (1, 2)\}$, then $B = B^{(0)}(B)$ and $-K^3(B) = 0$ (impossible). Thus $n_{1,r}^0 > 0$ for some $r \geq 3$. Notice that $\frac{r-1}{2r} \geq \frac{1}{3}$ for $r \geq 3$. It follows that $l(2) \geq \frac{3}{4} + \frac{1}{3} = \frac{13}{12}$. Thus we have $-K^3 \geq 2(P_{-1} - 3 + l(2)) \geq \frac{1}{6}$.

Subcase III-3. $P_{-2} = 3$. We have $\sigma = 3$. Also note that $P_{-4} \geq 2P_{-2} - 1 = 5$. By (4.1), we only need to consider the situation $P_{-4} \leq 6$.

Since $n_{1,3}^0 \geq 0$ and $n_{1,4}^0 \geq 0$, we have

$$3P_{-3} - 6 \geq P_{-4} \geq 2P_{-3} - 4.$$

Thus $(P_{-3}, P_{-4}) = (4, 5), (4, 6), (5, 6)$.

If $(P_{-3}, P_{-4}) = (5, 6)$, then $B^{(0)}(B) = \{3 \times (1, 3)\}$ and $B = B^{(0)}$ with $-K^3 = 0$, which is absurd.

If $(P_{-3}, P_{-4}) = (4, 6)$, then by 4.3, we may assume $\sigma_5 = 0$ and so that $B^{(0)}(B) = \{(1, 2), 2 \times (1, 4)\}$. Again $B = B^{(0)}(B)$ with $-K^3 = 0$, which is absurd.

If $(P_{-3}, P_{-4}) = (4, 5)$, then either $B^{(0)}(B) = \{(1, 2), (1, 3), (1, 4)\}$ or $B^{(0)}(B) = \{(1, 2), (1, 3), (1, s)\}$ with $s \geq 5$. For the first case, any packing of $B^{(0)}$ has negative $-K^3$. Thus the first case can not happen.

We consider the second case. If $s \geq 7$, then

$$-K^3(B) \geq -K^3(B^{(0)}(B)) = \frac{1}{42}.$$

If $s \leq 6$, we consider the one-step packing $B_1 = \{(2, 5), (1, s)\}$. Only when $s = 6$, $-K^3(B) \geq -K^3(B_1) = \frac{1}{30}$. This also means that $s = 5$ can not happen in this situation.

Case IV. $P_{-1} \geq 3$. If X is Gorenstein, then $-K^3 \geq 1$ since it is an integer. Otherwise, $-K^3 \geq 2l(2) \geq \frac{1}{2}$ by Riemann-Roch directly.

So we have proved the theorem. In fact, we have proved that $-K^3 = \frac{1}{330}$ if and only if $B = \{(1, 2), (2, 5), (1, 3), (2, 11)\}$. \square

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